

ON CARTAN JOINT SPECTRA

ENRICO BOASSO

ABSTRACT. In this work several results regarding the Cartan version of the Taylor, the Ślodkowski, the Fredholm, the split and the Fredholm split joint spectra will be studied.

KEYWORDS: Joint spectra, representations of solvable Lie algebras.

MSC (2000): Primary 47A13; Secondary 47A53, 47A80, 46M05, 17B30, 17B55.

1. INTRODUCTION

The main concern of the present work is noncommutative spectral theory within the Lie algebra framework. During the last years several joint spectra have been extended from commuting tuples of operators to representations of complex solvable finite dimensional Lie algebras in Banach spaces. Some of the spectra that have been considered are the Taylor, the Ślodkowski, the Fredholm, the split and the Fredholm split joint spectra, see for example the works [12], [5], [2], [16], [17] and [3]. As regard these spectra in the commuting case, see for instance [19], [18], [11], [14] and [9].

The most important difference between the nilpotent and the solvable Lie algebra setting consists in the so-called projection property. In fact, this property holds only for ideals in the solvable Lie algebra case, while it also holds for subalgebras when nilpotent Lie algebras are considered, see [12], [5], [2] and [17]. The problem of introducing a well-behaved joint spectrum in the solvable Lie algebra framework which extends the Taylor joint spectrum and at the same time preserves the projection property for subalgebras was solved in [1]. Moreover, this new version of the Taylor joint spectrum coincides with the one in [12], [5] and [16] for nilpotent Lie algebras, but it differs in general from the spectrum in these articles for solvable non nilpotent Lie subalgebras of operators, see [1] and [5]. Since in order to define the joint spectrum introduced in [1] Cartan subalgebras were considered, it was suggested that this spectrum should be called the Cartan-Taylor joint spectrum.

Furthermore, in [8] the Cartan-Ślodkowski joint spectra of a representation of a complex solvable finite dimensional Lie algebra in a Banach space were introduced. The joint spectra of [8] consist in the Cartan version of the Ślodkowski joint spectra and they provide a new extension of the spectra of [18] from commutative tuples of operators to representations of solvable Lie algebras in Banach spaces. In fact, as for the spectrum of [1], the joint spectra under consideration coincide with the ones of [2] and [17, 2.11] in the nilpotent Lie algebra case, but they differ in general from these spectra in the solvable non nilpotent Lie algebra setting, see [5] and [8]. Furthermore, as for the Cartan-Taylor joint spectrum,

the Cartan-Słodkowski joint spectra have the projection property for subalgebras even in the solvable setting.

The main concern of the present work consists in the study of several spectra related to the Cartan-Taylor and the Cartan-Słodkowski joint spectra. In fact, in section 2 the Cartan version of the Fredholm, the split and the Fredholm split joint spectra will be introduced and their main spectral properties will be proved. In addition, the relationships among these joint spectra will be considered. Furthermore, in section 3 all the aforesaid joint spectra of a solvable Lie algebra representation by compact operators in a Banach space will be described. Moreover, the behavior of all the above-mentioned joint spectra with respect to the procedure of passing from two representations of complex solvable finite dimensional Lie algebras in Banach spaces to the tensor product and the multiplication representation of the direct sum of the algebras will be studied. All these results will provide the Cartan version of several properties considered in [3] and [4].

The results of [3] were communicated in a talk given during the 20th International Conference on Operator Theory held in Timisoara, Rumania, from 30th June to 5th July 2004. It is a pleasure for the author to acknowledge the stimulating atmosphere and the warm hospitality of the Conference. Moreover, the author wish to express his fully indebtedness to the organizers of the Conference, specially to Luminita Stafi, Dan Timotin and Florian-Horia Vasilescu.

2. SPECTRA RELATED TO THE CARTAN-TAYLOR JOINT SPECTRUM

In this section the Cartan version of several joint spectra will be introduced. First of all, some notation is needed.

From now on, if X and Y are two Banach spaces, then $L(X, Y)$ denotes the algebra of all linear and continuous operators defined on X with values in Y , and $K(X, Y)$ the closed ideal of all compact operators of $L(X, Y)$. As usual, when $X = Y$, $L(X, X)$ and $K(X, X)$ are denoted by $L(X)$ and $K(X)$ respectively.

Consider a Banach space X , a complex solvable finite dimensional Lie algebra L , and a representation $\rho: L \rightarrow L(X)$. As regard the definitions and the main properties of the Taylor, the Słodkowski, the Fredholm, the split and the Fredholm split joint spectra of the representation ρ , see [12], [5], [2], [16], [17] and [3]. In the commutative framework, see [19], [18], [11], [14] and [9].

The following remark will be central in the definition of the Cartan spectra.

Remark 2.1. Let L be a complex solvable finite dimensional Lie algebra and H a nilpotent subalgebra of L . Consider the representation

$$ad_L: H \rightarrow \text{End}(L), \quad ad_L(h)(l) = [h, l],$$

where h and l belong to H and L respectively. Then, according to Theorem 1 of [6], there are finite linear functionals defined on H , α_i , $i = 1, \dots, m$, such that

$$L = \bigoplus_{i=1, \dots, m} L^{\alpha_i},$$

where $L^{\alpha_i} = \{l \in L: \text{there is } q \in \mathbb{N} \text{ such that } (ad_L(h) - \alpha_i(h))^q(l) = 0, \forall h \in H\}$. Furthermore, L^{α_i} are vector subspaces of L such that $ad_L(L^{\alpha_i}) \subseteq L^{\alpha_i}$, and $L^{\alpha_i} \neq 0$, $i = 1, \dots, m$.

Now well, since H is a nilpotent subalgebra of L , $H \subseteq L^0$, where L^0 denotes the subspaces defined by the null lineal functional of H . A nilpotent subalgebra is said a *Cartan subalgebra* of L if $H = L^0$. If L is a complex solvable finite dimensional Lie algebra, then L has at least one Cartan subalgebra, see Theorem 2 of [6] or Theorem 1, Chapter III, of [13].

As in [1] and in [8], given H a Cartan subalgebra of L , consider the Cartan decomposition of L determined by H , that is

$$L = H \oplus H_*,$$

where H_* is the direct sum of all the subspaces L^{α_i} , $i \in \llbracket 1, m \rrbracket$, such that $\alpha_i \neq 0$.

On the other hand, note that if L_1 and L_2 are two complex solvable finite dimensional Lie algebras, and if H_1 and H_2 are two Cartan subalgebras of L_1 and L_2 respectively, then a straightforward calculations shows that $H = H_1 \oplus H_2$ is a Cartan subalgebra of $L = L_1 \times L_2$, the direct sum of L_1 and L_2 . Furthermore, L can be decomposed as

$$L = H \oplus H_* = (H_1 \oplus H_2) \oplus (H_{1*} \oplus H_{2*}),$$

where $H_* = H_{1*} \oplus H_{2*}$ and

$$L_1 = H_1 \oplus H_{1*}, \quad L_2 = H_2 \oplus H_{2*}.$$

Next the definitions of the Cartan-Taylor and the Cartan-Słodkowski joint spectra will be reviewed, see [1] and [8]. Recall that if $\rho: L \rightarrow L(X)$ is a representation of the Lie algebra L in the Banach space X , then $\sigma(\rho)$, $\sigma_{\delta,k}(\rho)$ and $\sigma_{\pi,k}(\rho)$ denote respectively the Taylor and the Słodkowski joint spectra of ρ , see [12], [5], [2], [16] and [17].

Definition 2.2. *Let X be a complex Banach space, L a complex solvable finite dimensional Lie algebra, $\rho: L \rightarrow L(X)$ a representation of L in X , and H a Cartan subalgebra of L . Then, the Cartan-Taylor joint spectrum of ρ is the set*

$$\Sigma(\rho) = \{f \in L^* : f|_H \in \sigma(\rho|_H), \text{ and } f \text{ vanishes on } H_*\}.$$

In addition, the k -th δ -Cartan-Słodkowski joint spectrum of ρ is the set

$$\Sigma_{\delta,k}(\rho) = \{f \in L^* : f|_H \in \sigma_{\delta,k}(\rho|_H), \text{ and } f \text{ vanishes on } H_*\},$$

and the k -th π -Cartan-Słodkowski joint spectrum of ρ is the set

$$\Sigma_{\pi,k}(\rho) = \{f \in L^* : f|_H \in \sigma_{\pi,k}(\rho|_H), \text{ and } f \text{ vanishes on } H_*\},$$

where $\rho|_H: H \rightarrow L(X)$ is the restriction of ρ to the subalgebra H , $L = H \oplus H_$ is the Cartan decomposition of L , and $k = 0, \dots, n = \dim L$.*

Observe that

$$\Sigma_{\delta,n}(\rho) = \Sigma_{\pi,n}(\rho) = \Sigma(\rho).$$

Remark 2.3. First of all, note that according to [17, 2.4.4] and to [17, 0.5.8], the definition of the Cartan-Słodkowski joint spectra given above coincides with the one introduced in section 2 of [8].

Next, according to Lemma 2 of [8], the definition of each Cartan-Słodkowski joint spectrum is independent of the particular Cartan subalgebra of L used to define it. In addition, according to Lemma 1 of [8], each Cartan-Słodkowski joint

spectrum is contained in the sets of characters of L . As regard the Cartan-Taylor joint spectrum, see Theorems 2.1 and 2.6 of [1].

Furthermore, according to Theorem 1 of [8], each Cartan-Słodkowski joint spectrum has the projection property for Lie subalgebras of L , that is, if E is a subalgebra of L and if $\pi: L^* \rightarrow E^*$ denotes the restriction map, then

$$\pi(\Sigma_*(\rho)) = \Sigma_*(\rho|_E),$$

where Σ_* denotes one of the joint spectra in Definition 2.2 and $\rho|_E: E \rightarrow L(X)$ is the restriction of the representation ρ to the subalgebra E . As regard the Cartan-Taylor spectrum, see Theorem 2.6 of [1].

What is more, since the Słodkowski joint spectra are compact subsets of the set of characters of L , see [24, 2.11.3] or Theorems 4 and 7 of [2], a straightforward calculation shows that each Cartan-Słodkowski joint spectrum is a compact subset of the characters of L .

On the other hand, according to Theorem 2.6 of [1] and to the example that follows Definition 1 of [5], if L is a solvable not nilpotent Lie algebra, then $\Sigma_{\delta,k}(\rho)$ (resp. $\Sigma_{\pi,k}(\rho)$) differs in general from $\sigma_{\delta,k}(\rho)$ (resp. $\sigma_{\pi,k}(\rho)$), $k = 0, \dots, n$.

However, when L is a nilpotent Lie algebra

$$\Sigma_{\delta,k}(\rho) = \sigma_{\delta,k}(\rho), \quad \Sigma_{\pi,k}(\rho) = \sigma_{\pi,k}(\rho),$$

$k = 0, \dots, n$. In fact, in this case, since $H = L$ is a Cartan subalgebra of L , see [6] or Chapter 3 of [13], $H_* = 0$, and consequently the equalities are true.

Next the Cartan version of the Fredholm, the split and the Fredholm split joint spectra of a solvable Lie algebra representation in a Banach space will be introduced. Recall that if $\rho: L \rightarrow L(X)$ is a representation of the Lie algebra L in the Banach space X , then $\sigma_e(\rho)$, $\sigma_{\delta,k,e}(\rho)$ and $\sigma_{\pi,k,e}(\rho)$ denote the Fredholm joint spectra of ρ , see [3], $sp(\rho)$, $sp_{\delta,k}(\rho)$ and $sp_{\pi,k}(\rho)$ denote the split joint spectra of ρ , see [17] and [3], and $sp_e(\rho)$, $sp_{\delta,k,e}(\rho)$ and $sp_{\pi,k,e}(\rho)$ denote the Fredholm split joint spectra of ρ , see [3].

Definition 2.4. *Let X , L , $\rho: L \rightarrow L(X)$, and H be as in Definition 2.2. Then, the Cartan-Fredholm or Cartan-essential Taylor joint spectrum of ρ is the set*

$$\Sigma_e(\rho) = \{f \in L^*: f|_H \in \sigma_e(\rho|_H), \text{ and } f \text{ vanishes on } H_*\}.$$

In addition, the k -th Cartan-Fredholm or Cartan-essential δ -Słodkowski joint spectra of ρ is the set

$$\Sigma_{\delta,k,e}(\rho) = \{f \in L^*: f|_H \in \sigma_{\delta,k,e}(\rho|_H), \text{ and } f \text{ vanishes on } H_*\},$$

and the k -th Cartan-Fredholm or Cartan-essential π -Słodkowski joint spectrum of ρ is the set

$$\Sigma_{\pi,k,e}(\rho) = \{f \in L^*: f|_H \in \sigma_{\pi,k,e}(\rho|_H), \text{ and } f \text{ vanishes on } H_*\},$$

where $k = 0, \dots, n = \dim L$, and $\rho|_H: H \rightarrow L(X)$ is the restriction of the representation ρ to H .

Observe that $\Sigma_e(\rho) = \Sigma_{\delta,n,e}(\rho) = \Sigma_{\pi,n,e}(\rho)$.

Definition 2.5. Let $X, L, \rho: L \rightarrow L(X)$, and H be as in Definition 2.2. Then, the Cartan-split spectrum of ρ is the set

$$Sp(\rho) = \{f \in L^*: f|_H \in sp(\rho|_H), \text{ and } f \text{ vanishes on } H_*\}.$$

In addition, the k -th Cartan δ -split joint spectrum of ρ is the set

$$Sp_{\delta,k}(\rho) = \{f \in L^*: f|_H \in sp_{\delta,k}(\rho|_H), \text{ and } f \text{ vanishes on } H_*\},$$

and the k -th Cartan π -split joint spectrum of ρ is the set

$$Sp_{\pi,k}(\rho) = \{f \in L^*: f|_H \in sp_{\pi,k}(\rho|_H), \text{ and } f \text{ vanishes on } H_*\},$$

where $k = 0, \dots, n = \dim L$, and $\rho|_H: H \rightarrow L(X)$ is the restriction of the representation ρ to H .

Observe that $Sp_{\delta,n}(\rho) = Sp_{\pi,n}(\rho) = Sp(\rho)$.

Definition 2.6. Let $X, L, \rho: L \rightarrow L(X)$, and H be as in Definition 2.2. Then, the Cartan-Fredholm or Cartan-essential split spectrum of ρ is the set

$$Sp_e(\rho) = \{f \in L^*: f|_H \in sp_e(\rho|_H), \text{ and } f \text{ vanishes on } H_*\}.$$

In addition, the k -th Cartan-Fredholm or Cartan-essential δ -split joint spectrum of ρ is the set

$$Sp_{\delta,k,e}(\rho) = \{f \in L^*: f|_H \in sp_{\delta,k,e}(\rho|_H), \text{ and } f \text{ vanishes on } H_*\},$$

and the k -th Cartan-Fredholm or Cartan-essential π -split joint spectrum of ρ is the set

$$Sp_{\pi,k,e}(\rho) = \{f \in L^*: f|_H \in sp_{\pi,k,e}(\rho|_H), \text{ and } f \text{ vanishes on } H_*\},$$

where $k = 0, \dots, n = \dim L$, and $\rho|_H: H \rightarrow L(X)$ is the restriction of the representation ρ to H .

Observe that $Sp_{\delta,n,e}(\rho) = Sp_{\pi,n,e}(\rho) = Sp_e(\rho)$.

In what follows the properties of the sets introduced in definitions 2.4, 2.5 and 2.6 will be studied. In order to present the relationships among these sets and the Cartan-Taylor and the Cartan-Słodkowski joint spectra, some definitions will be reviewed.

As before, consider a complex solvable finite dimensional Lie algebra L , a complex Banach space X , and $\rho: L \rightarrow L(X)$ a representation of L in X . Then, if $C(X) = L(X)/K(X)$, $L_\rho: L \rightarrow L(L(X))$ and $\tilde{L}_\rho: L \rightarrow L(C(X))$ are the representations induced by left multiplication of operators in $L(X)$ and $C(X)$ respectively, which were studied in section 3 of [3] and [17].

On the other hand, L^{op} will denote the opposite algebra of L , that is the algebra that coincides with L as vector space and that has the bracket $[x, y]^{op} = -[x, y] = [y, x]$, where x and y belong to $L^{op} = L$ and $[\cdot, \cdot]$ is the Lie bracket of L . Then, if X' is the dual space of X , $\rho^*: L \rightarrow L(X')$ is the adjoint representation of ρ , see [2] and [17]. In addition, $R_\rho: L^{op} \rightarrow L(L(X))$ and $\tilde{R}_\rho: L^{op} \rightarrow L(C(X))$ are the representations induced by right multiplication of operators in $L(X)$ and $C(X)$ respectively, which were studied in [3] and [17].

Next denote by $l^\infty(X)$ the Banach space of all bounded sequences of elements of X with the supremum norm. Let $m(X)$ be the set of all sequences $(x_n)_{n \in \mathbb{N}}$

such that the closure of the set $\{x_n: n \in \mathbb{N}\}$ is compact. Then $m(X)$ is a closed subspace of $l^\infty(X)$. Denote $\tilde{X} = l^\infty(X)/m(X)$.

If X and Y are Banach spaces, and if $T \in L(X, Y)$, then T defines pointwise an operator $T^\infty: l^\infty(X) \rightarrow l^\infty(Y)$ by $T^\infty((x_n)_{n \in \mathbb{N}}) = (T(x_n))_{n \in \mathbb{N}}$. It is clear that $T^\infty(m(X)) \subseteq m(Y)$. Denote by $\tilde{T}: \tilde{X} \rightarrow \tilde{Y}$ the operator induced by T^∞ .

Now well, consider X , L and $\rho: L \rightarrow L(X)$ as before, and define the map

$$\tilde{\rho}: L \rightarrow L(\tilde{X}), \quad \tilde{\rho}(l) = \widetilde{\rho(l)},$$

where $l \in L$. According to Theorem 5 of [15], $\tilde{\rho}: L \rightarrow L(\tilde{X})$ is a representation of L in \tilde{X} , see also [7].

Theorem 2.7. *Let X be a complex Banach space, L a complex solvable finite dimensional Lie algebra, and $\rho: L \rightarrow L(X)$ a representation of L in X . Then,*

- i) $\Sigma_{\pi,k}(\rho) = \Sigma_{\delta,k}(\rho^*)$, $\Sigma_{\delta,k}(\rho) = \Sigma_{\pi,k}(\rho^*)$, $\Sigma(\rho) = \Sigma(\rho^*)$,
- ii) $\Sigma_{\pi,k,e}(\rho) = \Sigma_{\delta,k,e}(\rho^*)$, $\Sigma_{\delta,k,e}(\rho) = \Sigma_{\pi,k,e}(\rho^*)$, $\Sigma_e(\rho) = \Sigma_e(\rho^*)$,
- iii) $\Sigma_{\delta,k,e}(\rho) = \Sigma_{\delta,k}(\tilde{\rho})$, $\Sigma_{\pi,k,e}(\rho) = \Sigma_{\pi,k}(\tilde{\rho})$, $\Sigma_e(\rho) = \Sigma(\tilde{\rho})$,
- iv) $\Sigma_{\pi,k,e}(\rho) = \Sigma_{\delta,k}((\tilde{\rho})^*)$, $\Sigma_e(\rho) = \Sigma((\tilde{\rho})^*)$,
- v) $Sp_{\delta,k}(\rho) = \Sigma_{\delta,k}(L_\rho)$, $Sp_{\pi,k}(\rho) = \Sigma_{\delta,k}(R_\rho)$, $Sp(\rho) = \Sigma(L_\rho) = \Sigma(R_\rho)$,
- vi) $Sp_{\delta,k,e}(\rho) = \Sigma_{\delta,k}(\tilde{L}_\rho)$, $Sp_{\pi,k,e}(\rho) = \Sigma_{\delta,k}(\tilde{R}_\rho)$, $Sp_e(\rho) = \Sigma(\tilde{L}_\rho) = \Sigma(\tilde{R}_\rho)$,

where $k = 0, \dots, n = \dim L$.

Proof. It is a consequence of Definitions 2.2, 2.4, 2.5 and 2.6, Theorem 7 of [2], [17, 2.11.4], Theorem 4 of [3], [17, 3.1.5], [17, 3.1.7], Theorem 8 of [3], Proposition 3.1 of [7], and [17, 0.5.8]. \square

Theorem 2.8. *Let X be a complex Banach space, L a complex solvable finite dimensional Lie algebra, and $\rho: L \rightarrow L(X)$ a representation of L in X . Then,*

i) *the sets $\Sigma_e(\rho)$, $\Sigma_{\delta,k,e}(\rho)$, $\Sigma_{\pi,k,e}(\rho)$, $Sp(\rho)$, $Sp_{\delta,k}(\rho)$, $Sp_{\pi,k}(\rho)$, $Sp_e(\rho)$, $Sp_{\delta,k,e}(\rho)$, and $Sp_{\pi,k,e}(\rho)$ are independent of the particular Cartan subalgebra of L used to define them,*

ii) *all the sets considered in i) are compact nonempty subsets of the character of L that have the projection property for Lie subalgebras of L .*

Proof. It is a consequence of Theorem 2.7 and Lemma 1, Lemma 2 and Theorem 1 of [8], see also Proposition 2.1 and Theorem 2.6 of [1]. \square

Remark 2.9. In the above conditions, it is clear that

$$\begin{aligned} \Sigma_{\delta,k}(\rho) &\subseteq Sp_{\delta,k}(\rho), & \Sigma_{\pi,k}(\rho) &\subseteq Sp_{\pi,k}(\rho), & \Sigma(\rho) &\subseteq Sp(\rho), \\ \Sigma_{\delta,k,e}(\rho) &\subseteq Sp_{\delta,k,e}(\rho), & \Sigma_{\pi,k,e}(\rho) &\subseteq Sp_{\pi,k,e}(\rho), & \Sigma_e(\rho) &\subseteq Sp_e(\rho). \end{aligned}$$

Furthermore, if X is a Hilbert space, then the above inclusions are equalities.

In addition, according to Theorem 2.6 of [1] and to the example that follows Definition 1 of [5], note that if L is a solvable not nilpotent Lie algebra, then

$Sp_{\delta,k}(\rho)$ (resp. $Sp_{\pi,k}(\rho)$) differs in general from $sp_{\delta,k}(\rho)$ (resp. $sp_{\pi,k}(\rho)$), $k = 0, \dots, n$.

However, when L is a nilpotent Lie algebra

- i) $\Sigma_{\delta,k,e}(\rho) = \sigma_{\delta,k,e}(\rho)$, $\Sigma_{\pi,k,e}(\rho) = \sigma_{\pi,k,e}(\rho)$,
- ii) $Sp_{\delta,k}(\rho) = sp_{\delta,k}(\rho)$, $Sp_{\pi,k}(\rho) = sp_{\pi,k}(\rho)$,
- iii) $Sp_{\delta,k,e}(\rho) = sp_{\delta,k,e}(\rho)$, $Sp_{\pi,k,e}(\rho) = sp_{\pi,k,e}(\rho)$,

where $k = 0, \dots, n$. In fact, in this case, since $H = L$ is a Cartan subalgebra of L , see [6] or Chapter 3 of [13], $H_* = 0$, and consequently the equalities are true.

3. PROPERTIES OF THE JOINT SPECTRA

In this section several results regarding the joint spectra of Definitions 2.2 and 2.4 - 2.6 will be studied. In first place, solvable Lie algebra representations by compact operators in Banach spaces will be considered both in the infinite and finite dimensional case.

Theorem 3.1. *Let X be an infinite dimensional complex Banach space, L a complex solvable finite dimensional Lie algebra, and $\rho: L \rightarrow L(X)$ a representation of L in X , such that $\rho(l) \in K(X)$ for each $l \in L$. In addition, suppose that H is a Cartan subalgebra of L . Then,*

- i) $\Sigma(\rho) = Sp(\rho)$, $\Sigma_{\delta,k}(\rho) = Sp_{\delta,k}(\rho)$ and $\Sigma_{\pi,k}(\rho) = Sp_{\pi,k}(\rho)$,
- ii) the sets $\Sigma(\rho)$, $\Sigma_{\delta,k}(\rho) \cup \{0\}$, $\Sigma_{\pi,k}(\rho) \cup \{0\}$, $Sp(\rho)$, $Sp_{\delta,k}(\rho) \cup \{0\}$ and $Sp_{\pi,k}(\rho) \cup \{0\}$ coincide with the set

$$\{0\} \cup \{f \in L^* : f(L^2) = 0, \text{ such that there is } x \in X, x \neq 0, \\ \text{with the property } \rho(h)x = f(h)x, \forall h \in H\},$$

- iii) $\Sigma_e(\rho) = \Sigma_{\delta,k,e}(\rho) = \Sigma_{\pi,k,e}(\rho) = \{0\}$,
 - iv) $sp_e(\rho) = sp_{\delta,k,e}(\rho) = sp_{\pi,k,e}(\rho) = \{0\}$,
- where $k = 0, \dots, n = \dim L$.

Proof. It is a consequence of Definitions 2.2, 2.4, 2.5 and 2.6, and Theorems 1, 2 and 3 of [4]. \square

In the conditions of Theorem 3.1, note that when X is finite dimensional, the spectra to be considered are the Cartan-Taylor and the Cartan-Słodkowski joint spectra.

Theorem 3.2. *Let X be a complex finite dimensional Banach space, L a complex solvable finite dimensional Lie algebra, and $\rho: L \rightarrow L(X)$ a representation of L in X . In addition, suppose that H a Cartan subalgebra of L . Then,*

$$\Sigma(\rho) = \Sigma_{\delta,k}(\rho) = \Sigma_{\pi,k}(\rho) = \{f \in L^* : f(L^2) = 0, \text{ such that there is } \\ x \in X, x \neq 0, \text{ with the property: } \rho(h)x = f(h)x, \forall h \in H\}.$$

Proof. It is a consequence of Definition 2.2 and Theorem 4 of [4]. \square

Next the spectral contributions of two representations of complex solvable finite dimensional Lie algebras in Banach spaces to the Cartan-Taylor, the Cartan-Słodkowski, the Cartan-Fredholm, the Cartan-split and the Cartan-Fredholm split joint spectra of the tensor product and the multiplication representations of the direct sum of the algebras will be described. The results to be presented consist in an extension to these spectra of the ones obtained in [3].

In what follows the axiomatic tensor product introduced in [10] will be considered. As regard its definition as well as the definition of the class of ideals between Banach spaces to be used, see [10] and [3]. Furthermore, in order to review the definition of the tensor product and the multiplication representation, see [10], [17] and [3]. Recall that if X is a Banach space, then X' will denote the dual space of X .

Theorem 3.3. *Let X_1 and X_2 be two complex Banach spaces, L_1 and L_2 two complex solvable finite dimensional Lie algebras, and $\rho_i: L_i \rightarrow L(X_i)$, $i = 1, 2$, two representations of the Lie algebras. Suppose that $X_1 \tilde{\otimes} X_2$ is a tensor product of X_1 and X_2 relative to $\langle X_1, X_1' \rangle$ and $\langle X_2, X_2' \rangle$, and consider the tensor product representation of $L = L_1 \times L_2$, $\rho: L \rightarrow L(X_1 \tilde{\otimes} X_2)$. Then,*

$$i) \bigcup_{p+q=k} \Sigma_{\delta,p}(\rho_1) \times \Sigma_{\delta,q}(\rho_2) \subseteq \Sigma_{\delta,k}(\rho) \subseteq Sp_{\delta,k}(\rho) \subseteq \bigcup_{p+q=k} Sp_{\delta,p}(\rho_1) \times Sp_{\delta,q}(\rho_2),$$

$$ii) \bigcup_{p+q=k} \Sigma_{\pi,p}(\rho_1) \times \Sigma_{\pi,q}(\rho_2) \subseteq \Sigma_{\pi,k}(\rho) \subseteq Sp_{\pi,k}(\rho) \subseteq \bigcup_{p+q=k} Sp_{\pi,p}(\rho_1) \times Sp_{\pi,q}(\rho_2).$$

In particular, if X_1 and X_2 are Hilbert spaces, the above inclusions are equalities.

Proof. It is a consequence of Definitions 2.2 and 2.5, Theorem 14 of [3] and Remark 2.1. \square

Theorem 3.4. *In the same conditions of Theorem 3.3,*

$$i) \bigcup_{p+q=k} \Sigma_{\delta,p,e}(\rho_1) \times \Sigma_{\delta,q}(\rho_2) \bigcup_{p+q=k} \Sigma_{\delta,p}(\rho_1) \times \Sigma_{\delta,q,e}(\rho_2) \subseteq \sigma_{\delta,k,e}(\rho) \subseteq Sp_{\delta,k,e}(\rho) \subseteq \bigcup_{p+q=k} Sp_{\delta,p,e}(\rho_1) \times Sp_{\delta,q}(\rho_2) \bigcup_{p+q=k} Sp_{\delta,p}(\rho_1) \times Sp_{\delta,q,e}(\rho_2),$$

$$\begin{aligned}
ii) \quad & \bigcup_{p+q=k} \Sigma_{\pi,p,e}(\rho_1) \times \Sigma_{\pi,q}(\rho_2) \bigcup \bigcup_{p+q=k} \Sigma_{\pi,p}(\rho_1) \times \Sigma_{\pi,q,e}(\rho_2) \subseteq \Sigma_{\pi,k,e}(\rho) \subseteq \\
& Sp_{\pi,k,e}(\rho) \subseteq \bigcup_{p+q=k} Sp_{\pi,p,e}(\rho_1) \times Sp_{\pi,q}(\rho_2) \bigcup \bigcup_{p+q=k} Sp_{\pi,p}(\rho_1) \times Sp_{\pi,q,e}(\rho_2).
\end{aligned}$$

In particular, if X_1 and X_2 are Hilbert spaces, the above inclusions are equalities.

Proof. It is a consequence of Definitions 2.4 and 2.6, Theorem 17 of [3], and Remark 2.1 \square

Theorem 3.5. *Let X_1 and X_2 be two complex Banach spaces, L_1 and L_2 two complex solvable finite dimensional Lie algebras, and $\rho_i: L_i \rightarrow L(X_i)$, $i = 1, 2$, two representations of the Lie algebras. Suppose that J is an operator ideal between X_2 and X_1 , and present it as the tensor product of X_1 and X_2' relative to $\langle X_1, X_1' \rangle$ and $\langle X_2', X_2 \rangle$. In addition, consider the multiplication representation of $L = L_1 \times L_2^{op}$, $\tilde{\rho}: L \rightarrow L(J)$. Then, if $\dim L_2 = m$,*

$$\begin{aligned}
i) \quad & \bigcup_{p+q=k} \Sigma_{\delta,p}(\rho_1) \times \Sigma_{\pi,m-q}(\rho_2) \subseteq \Sigma_{\delta,k}(\tilde{\rho}) \subseteq Sp_{\delta,k}(\tilde{\rho}) \subseteq \\
& \bigcup_{p+q=k} Sp_{\delta,p}(\rho_1) \times Sp_{\pi,m-q}(\rho_2),
\end{aligned}$$

$$\begin{aligned}
ii) \quad & \bigcup_{p+q=k} \Sigma_{\pi,p}(\rho_1) \times \Sigma_{\delta,m-q}(\rho_2) \subseteq \Sigma_{\pi,k}(\tilde{\rho}) \subseteq Sp_{\pi,k}(\tilde{\rho}) \subseteq \\
& \bigcup_{p+q=k} Sp_{\pi,p}(\rho_1) \times Sp_{\delta,m-q}(\rho_2).
\end{aligned}$$

In particular, if X_1 and X_2 are Hilbert spaces, the above inclusions are equalities.

Proof. It is a consequence of Definitions 2.2 and 2.5, Theorem 19 of [3], [17, 0.5.8], and Remark 2.1. \square

Theorem 3.6. *In the same conditions of Theorem 3.5,*

$$\begin{aligned}
i) \quad & \bigcup_{p+q=k} \Sigma_{\delta,p,e}(\rho_1) \times \Sigma_{\pi,m-q}(\rho_2) \bigcup \bigcup_{p+q=k} \Sigma_{\delta,p}(\rho_1) \times \Sigma_{\pi,m-q,e}(\rho_2) \subseteq \\
& \Sigma_{\delta,k,e}(\tilde{\rho}) \subseteq Sp_{\delta,k,e}(\tilde{\rho}) \subseteq \\
& \bigcup_{p+q=k} Sp_{\delta,p,e}(\rho_1) \times Sp_{\pi,m-q}(\rho_2) \bigcup \bigcup_{p+q=k} Sp_{\delta,p}(\rho_1) \times Sp_{\pi,m-q,e}(\rho_2), \\
ii) \quad & \bigcup_{p+q=k} \Sigma_{\pi,p,e}(\rho_1) \times \Sigma_{\delta,m-q}(\rho_2) \bigcup \bigcup_{p+q=k} \Sigma_{\pi,p}(\rho_1) \times \Sigma_{\delta,m-q,e}(\rho_2) \subseteq \\
& \Sigma_{\pi,k,e}(\tilde{\rho}) \subseteq Sp_{\delta,k,e}(\tilde{\rho}) \subseteq \\
& \bigcup_{p+q=k} Sp_{\pi,p,e}(\rho_1) \times Sp_{\delta,m-q}(\rho_2) \bigcup \bigcup_{p+q=k} Sp_{\pi,p}(\rho_1) \times Sp_{\delta,m-q,e}(\rho_2).
\end{aligned}$$

In particular, if X_1 and X_2 are Hilbert spaces, the above inclusions are equalities.

Proof. It is a consequence of Definition 2.4 and 2.6, Theorem 20 of [3], [17, 0.5.8], and Remark 2.1. \square

A final remark. It is worth noticing that as in the previous results, the analogues of Theorems 23 and 24 of [3] can be also proved for the joint spectra of Definitions 2.2 and 2.4 - 2.6.

REFERENCES

- [1] D. Beltita, Spectrum for a solvable Lie algebra of operators, *Studia Math.* 135 (1999), 163-178.
- [2] E. Boasso, Dual properties and joint spectra for solvable Lie algebras of operators, *J. Operator Theory* 33 (1995), 105-116.
- [3] E. Boasso, Joint spectra of the tensor product representation of the direct sum of two solvable Lie algebras, *Dissertationes Math.* 416 (2003).
- [4] E. Boasso, Joint spectra of representations of Lie algebras by compact operators, *Glasgow Math. J.* 46 (2004), 355-362.
- [5] E. Boasso, A. Larotonda, A spectral theory for solvable Lie algebras of operators, *Pacific J. Math.* 158 (1993), 15-22.
- [6] F. Bruhat, Poids et racines I, *Séminaire Sophus Lie*, 1e année, 1954-1955, Théorie des algèbres de Lie. Topologie de groupes de Lie, Exposé 9, Secrétariat Mathématique, Paris (1955).
- [7] A. Dosiev, Taylor spectrum of a representation of a Banach-Lie algebra, Ph. D. Baku State University, Baku (1998).
- [8] A. Dosiev, Projection property of spectra for solvable Lie algebra representations, *Trans. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci.* (239 (2003), 27-32.
- [9] J. Eschmeier, Analytic spectral mapping theorems for joint spectra. Operators in indefinite metric spaces, scattering theory and other topics (Bucharest, 1985), *Oper. Theory Adv. Appl.*, Birkhäuser, Basel vol. 24, (1987), 167-181.
- [10] J. Eschmeier, Tensor products and elementary operators, *J. Reine Angew. Math.* 390 (1988), 47-66.

- [11] A. S. Fainshtein, Joint essential spectrum of a family of linear operators, *Funct. Anal. Appl.* 14 (1980), 152-153.
- [12] A. S. Fainshtein, Taylor joint spectrum for families of operators generating nilpotent Lie algebras, *J. Operator Theory* 29 (1993), 3-27.
- [13] N. Jacobsen, *Lie Algebras*, Interscience Publishers, New York and London, 1962.
- [14] V. Müller, The Ślodkowski spectra and higher Shilov Boundaries, *Studia Math.* 105 (1993), 69-75.
- [15] V. Müller, Stability of index for semi-Fredholm chains, *J. Operator Theory* 37 (1997) 247-26.
- [16] C. Ott, A note on a paper of E. Boasso and A. Larotonda, *Pacific J. Math.* 173 (1996), 173-179.
- [17] C. Ott, Gemeinsame Spektren auflösbarer Operator-Liealgebren, Dissertation, Kiel, 1997 (<http://analysis.math.uni-kiel.de/wrobel/>).
- [18] Z. Ślodkowski, An infinite family of joint spectra, *Studia Math.* 61 (1973), 239-255.
- [19] J. L. Taylor, A joint spectrum for several commuting operators, *J. Funct. Anal.* 6 (1970), 172-191.

Enrico Boasso

E-mail address: enrico_odisseo@yahoo.it